Multiscale finite element methods for miscible and immiscible flow in porous media

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1 Introduction

Flow and transport in fractured porous media are, very often, processes *not* dominated by diffusion. This makes the mathematical problem almost hyperbolic, which naturally develops sharp features in the solution. Classical numerical methods produce a solution that either lacks *stability*, resulting in nonphysical oscillations, or *accuracy*, by showing excessive numerical diffusion.

Development of novel numerical methods for the complete equations of multiphase compositional flow in multidimensions must necessarily start from simplified models in one space dimension. These reduced model problems should display, however, the key features which pose difficulties in obtaining satisfactory numerical solutions such as, for instance, wild nonlinearity, shocks or near-shocks, boundary layers and degenerate diffusion. The key point of the proposed formulation is a multiscale decomposition of the variable of interest into resolved (or grid) scales and unresolved (or subgrid) scales, which acknowledges the fact that the fine-scale structure of the solution cannot be captured by *any* mesh. However, the influence of the subgrid scales on the resolvable scales is not negligible. A novel idea of subgrid stabilization by means of the concave hull of the flux function is introduced. By accounting for the subgrid scales, the oscillatory behavior of classical Galerkin is drastically reduced and confined to a small neighborhood containing the sharp features, while the solution is high-order accurate where the solution is smooth. This ensures that the numerical solution is not globally deteriorated. The method does *not* emanate from a monotonicity argument and, therefore, it does not rule out small overshoots and undershoots near the sharp layers. To prevent this situation, a subscale-driven shock-capturing mechanism is presented. The generality of the proposed formulation makes it amenable to further extensions.

2 Numerical formulation: the multiscale approach

We model miscible and immiscible flow in porous media with the scalar conservation law:

$$\partial_t u + \nabla \cdot \boldsymbol{\sigma} = q, \quad x \in \Omega,\ t \in (0, T], \tag{1}$$

where u is the conserved quantity, $\boldsymbol{\sigma}$ is the total flux, and q is the rate of production. In the nonlinear case, the total flux has the form $\boldsymbol{\sigma} = \boldsymbol{f}(u) - \mathbf{k}(u)\nabla u$, where $\boldsymbol{f}$ is the hyperbolic part of the flux and $\mathbf{k}$ is the diffusion tensor. Both are allowed to be nonlinear functions of the unknown u. We consider Dirichlet and Neumann boundary conditions:

$$u = \bar{u} \text{ on } \Gamma_u \subset \partial\Omega, \qquad \boldsymbol{\sigma} \cdot \boldsymbol{n} = \bar{\sigma} \text{ on } \Gamma_\sigma = \partial\Omega \setminus \Gamma_u, \tag{2}$$

and the initial conditions $u(x,t=0)=u_0(x)$. In the linear case, $\boldsymbol{\sigma}=\boldsymbol{a}u-\mathbf{k}\nabla u$, and we define the linear advection-diffusion operator $\mathcal{L}u := \nabla\cdot(\boldsymbol{a}u-\mathbf{k}\nabla u)$.

Consider the functional spaces $\mathcal{V} := \{v\in W : v=\bar{u} \text{ on } \Gamma_u\}$, $\mathcal{V}_0 := \{v\in W : v=0 \text{ on } \Gamma_u\}$, where the appropriate choice of the Sobolev space W depends on the form of the diffusion tensor. The **weak form** of problem is to seek $u\in\mathcal{V}$ for each fixed $t\in(0,T]$, such that

$$(\partial_t u, v) + a(u,v;u) = l(v) \quad \forall v\in\mathcal{V}_0, \qquad u(x,t=0)=u_0(x), \tag{3}$$

where $(\partial_t u, v)=\int_\Omega \partial_t u\, v\, dx$, $a(u,v;w) = -\int_\Omega \boldsymbol{f}(w)\cdot\nabla v\, d\Omega + \int_\Omega \mathbf{k}(w)\nabla u\cdot\nabla v\, d\Omega$, and $l(v)=\int_\Omega qv\, d\Omega - \int_{\Gamma_\sigma}\bar{\sigma}v\, d\Gamma$. In the linear case the only difference with respect to Equation (3) is that $a(u,v)\equiv a(u,v;u)$ is now a *bilinear* form.

Let $\mathcal{V}_h\subset\mathcal{V}$, and $\mathcal{V}_{h,0}\subset\mathcal{V}_0$, be *conforming* finite element spaces of piecewise polynomials. The **standard Galerkin** approximation of (3) is to find $u_h\in\mathcal{V}_h$ for each fixed t, such that

$$(\partial_t u_h, v_h) + a(u_h,v_h;u_h) = l(v_h) \quad \forall v_h\in\mathcal{V}_{h,0}, \tag{4}$$

and $u_h(x,t=0)$ is the projection of the initial function $u_0(x)$ onto space $\mathcal{V}_h$. The standard Galerkin method lacks stability for the near-hyperbolic problem, as shown in the next section.

The *key idea* of the **multiscale formulation** [2] is to consider $\mathcal{V}$ and $\mathcal{V}_0$ as the direct sum of two spaces,

$$\mathcal{V}=\mathcal{V}_h\oplus\tilde{\mathcal{V}}, \qquad \mathcal{V}_0=\mathcal{V}_{h,0}\oplus\tilde{\mathcal{V}}, \tag{5}$$

where $\mathcal{V}_h$ and $\mathcal{V}_{h,0}$ are the spaces of *resolved scales* and $\tilde{\mathcal{V}}$ is the space of *subgrid scales*. This decomposition acknowledges the fact that the subscales cannot be captured by the finite element mesh, but their influence on the grid scales is *not* negligible. We can now split the original **linear problem**:

$$\text{Grid scales: } (\partial_t(u_h+\tilde{u}), v_h) + a(u_h+\tilde{u}, v_h) = l(v_h) \quad \forall v_h\in\mathcal{V}_{h,0}, \tag{6}$$

$$\text{Subscales: } (\partial_t(u_h+\tilde{u}), \tilde{v}) + a(u_h+\tilde{u}, \tilde{v}) = l(\tilde{v}) \quad \forall \tilde{v}\in\tilde{\mathcal{V}}. \tag{7}$$

The subscales are modeled analytically using an algebraic subgrid scale (ASGS) approximation [1], $\tilde{u}\approx\tau\mathcal{R}u_h$, where τ is called the *relaxation time* and $\mathcal{R}u_h := q-\partial_t u_h-\mathcal{L}u_h$ is the *grid scale residual*. After integration by parts on each element, the equation for the grid scales reads

$$(\partial_t u_h, v_h) + a(u_h,v_h) + \sum_e\left[\int_{\Omega^e}\tilde{u}\,\mathcal{L}^* v_h\, d\Omega + \int_{\Gamma^e}\tilde{u}\, b^* v_h\, d\Gamma\right] = l(v_h) \quad \forall v_h\in\mathcal{V}_{h,0}, \tag{8}$$

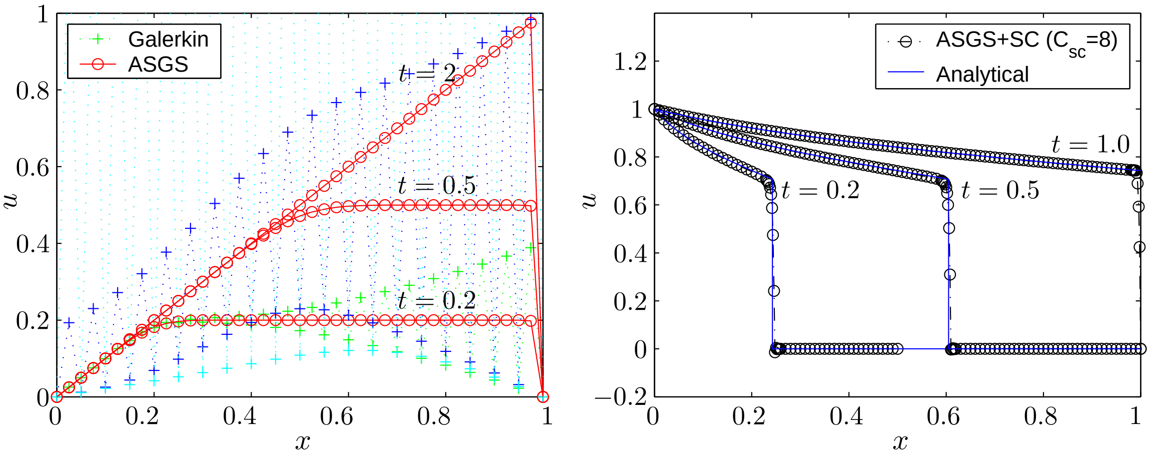
where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$, and b^* is the associated boundary operator. The final equation for the resolved scales includes the usual Galerkin terms and some additional volume and boundary integrals evaluated element by element. Since the subscales $\tilde{u}$ are proportional to the grid scale residual, the method is residual-based and, therefore, automatically consistent.

Application of the multiscale approach to the **nonlinear problem** is not straightforward because the form $a(u,v;w)$ is *not* linear in w. Our approach is based on an incremental formulation and a multiple scale decomposition of the increment: $u\approx u_h+\delta\tilde{u}$. This allows us to split the problem into a grid-scale problem and a subscale problem. The final equation are formally identical to those of the linear case, but now involve a linearized advection-diffusion operator. During the course of this investigation, we have seen that “classical” stabilized methods fail to produce a satisfactory solution to nonlinear problems with nonconvex flux function. We use the concave hull of the flux function *for the subgrid scales only*. The motivation is to eliminate the residual part of the flux function, which counterbalances diffusion. We further improve the method by incorporating a *shock-capturing* technique, based on increasing the amount of numerical dissipation in the neighborhood of layers. We propose a novel dimensionally-consistent, *subscale-driven* artificial diffusion, given by $k_{sc}=C_{sc}(|\delta\tilde{u}|/U_{sc})h|\boldsymbol{a}(u_h)|$, where U_{sc} is a characteristic value of the solution near the shock, and C_{sc} is a constant coefficient.

3 Representative simulations

Under certain simplifying assumptions, the case of one-dimensional **miscible flow** corresponds to a linear advection-diffusion equation, with $f(u) = au$, $k(u) = \epsilon$. We solve the problem with parameters: $a = 1$, $\epsilon = 10^{-4}$, $q = 1$. A backward Euler time-stepping scheme and a fairly coarse grid (the element Peclet number is 250) was used. Standard Galerkin method produces a globally oscillatory solution at all times, while the solution obtained by the ASGS method is nonoscillatory and captures sharply the moving ramp-plateau interface and the boundary layer (Figure (a)).

One-dimensional **immiscible flow** of two fluids is described by the Buckley-Leverett equation. Here we employ $f(u) = q_T u^2/(u^2 + (1-u)^2)$ and $k(u) = \epsilon u(1-u)$ with values: $q_T = 1$, $\epsilon = 10^{-4}$. We used backward Euler time-stepping and different grids (for the results shown the element Peclet number is about 100). Classical Galerkin gives a globally oscillatory solution, especially for long simulation times. The subgrid scale approach, in particular when shock-capturing is included, provides a solution that is free from oscillations and not overly diffusive (Figure (b)).



(a) Linear advection-diffusion with sources

(b) Buckley-Leverett flow

4 Conclusions

We have presented a successful numerical method for the solution of nonlinear conservation laws, which is based on a multiscale decomposition of the variable of interest, and applied it to the problems of miscible and immiscible two-phase flow in porous media. To the best of our knowledge, this approach is entirely new in the context of flow in porous media. The proposed modified subgrid scale method with shock-capturing shows exceptional performance in the test cases studied. We are now extending this methodology to multiphase compositional flows in multidimensions.

References

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